



A Consistent Partly and Fully Cracked XFEM Element for Modelling Fracture in Concrete Structures

Mougaard, Jens Falkenskov; Poulsen, Peter Noe; Nielsen, Leif Otto

Published in:
Computational Technologies in Concrete Structures

Publication date:
2009

Document Version
Publisher's PDF, also known as Version of record

[Link back to DTU Orbit](#)

Citation (APA):
Mougaard, J. F., Poulsen, P. N., & Nielsen, L. O. (2009). A Consistent Partly and Fully Cracked XFEM Element for Modelling Fracture in Concrete Structures. In *Computational Technologies in Concrete Structures*

General rights

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

A Consistent Partly and Fully Cracked XFEM Element for Modelling Fracture in Concrete Structures

J. F. Mougaard¹⁾, P. N. Poulsen²⁾ and L. O. Nielsen²⁾

^{1),2)}*Department of Civil Engineering, DTU, Lyngby, Denmark*

¹⁾ *jfm@byg.dtu.dk*

ABSTRACT

This paper discusses the build-up of a partly cracked cohesive crack-tip element of higher order. The crack tip element is based on the principles of the eXtended Finite Element Method (XFEM) and is of Linear Strain Triangle (LST) type. The composition of the enrichment has been in focus in order to achieve as complete a description as possible on both sides of the crack. The stress accuracy within the crack tip element has been improved to a level, so that the crack tip stresses can be evaluated locally without introducing nonlocal averaging in the surroundings of the crack tip. The partly cracked element can with a few restrictions in the displacement field be applied as a fully cracked element. The performance of the developed element has been tested against a semianalytical solution of an infinite sheet with an initial flaw in pure tension and performs well.

INTRODUCTION

XFEM has shown good results when modelling cohesive crack growth Belytschko and Black (2003). When modelling cohesive crack growth it is for a number of reasons important to have partly cracked elements. In order to write a real incremental form of the FE-equations a partly cracked element is needed. In general all cracked stages should be possible to describe, so that all points on a response curve can be evaluated and not only points corresponding to fully cracked stages. In general a discretization with as few elements as possible is preferred. With a rough discretization the partly cracked elements are essential in order to determine the solution between fully cracked stages.

In a previous work a partly cracked XFEM element with an appealingly simple displacement field has been suggested Zi and Belytschko (2003). However as for the fully cracked element only one discontinuity DOF is active when considering the Constant Strain Triangle (CST) element. The element is therefore not able to model equal stresses on both sides of the crack. Furthermore the chosen geometrical position of the discontinuous enrichment triangle in the element gives a

¹⁾Ph.D. student and Director

²⁾Associate Professors

somewhat skew stiffness distribution. This results in some zigzag behaviour on the load deflection diagram.

A more complete displacement field is proposed by Asferg et al. (2007). This displacement field has an extra enrichment node, so that the element is capable of modelling equal stresses on both sides of the crack. A generalization to higher order elements has turned out to be difficult.

The present work introduces two nearly symmetrically placed enrichment fields (double enriched displacement fields) in higher order triangular elements. The presented work shows an implementation of the LST-type. Traditionally the enrichment fields in partly cracked elements has been of discontinuous type only. When the element is partly cracked these discontinuous enrichments are not sufficient and gives an incomplete displacement field in the partly cracked region of the element. This is improved by adding both continuous and discontinuous enrichment fields in the element. The partly cracked element can as well be used as a fully cracked element. This improved crack tip element has been implemented, and shows a rather precise and smooth response diagram.

DISPLACEMENT FIELD

The basic concept in XFEM is to introduce a discontinuous enrichment field besides the standard continuous field. Therefore it is natural to write the complete field as a sum of a continuous and a discontinuous contribution as shown below

$$\mathbf{u}(x, y) = \mathbf{N}_c(x, y)\mathbf{v}_c + \mathbf{N}_d(x, y)\mathbf{v}_d \quad (1)$$

The discontinuous enrichment is chosen of same shape as the continuous displacement field

$$\mathbf{N}_d(x, y) = \sum_I \mathbf{N}_{dI}(x, y) \quad \text{where} \quad \mathbf{N}_{dI}(x, y) = H_I(x, y)\mathbf{N}_{cI}(x, y) \quad (2)$$

Here $H_I(x, y)$ is the 2D Heaviside step function defined for each enrichment node in the set I . These Heaviside step functions is defined as zero on the same side of the discontinuity as the enrichment nodes and 1 on the opposite side of the discontinuity.

This concept applies per definition for fully cracked elements, where two completely independent displacement fields can be achieved on both sides of the crack. Meanwhile the enrichment of a partly cracked element is not as trivial. Applying a single discontinuous enrichment field in the partly cracked element leaves a limited number of degrees of freedom to handle the rather complex stress field in the near surroundings of the crack tip. Therefore a more detailed enrichment is needed in the partly cracked element.

Double enriched displacement field

The element should be capable of representing equal stresses on both sides of the crack at the tip. This is achieved with an additional enrichment in the tip element. The displacement field then consist of the standard field from the element triangle A and two enrichments from the subtriangles B and C. The concept is illustrated in Figure 1.

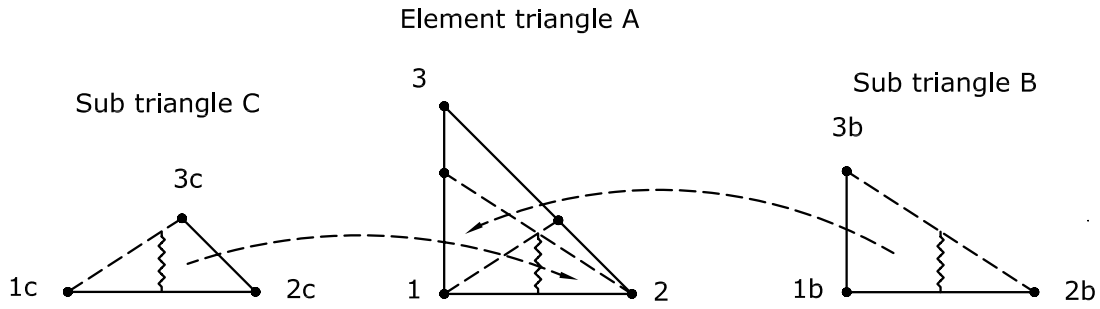


Figure 1: Composition of the enrichments in the tip element

In order to achieve a complete enrichment in both subtriangles it is necessary not only to include the basic discontinuous displacement fields normally used in XFEM, but also continuous contributions in the subtriangles. This allows the element to describe the conditions with equal stresses on the crack faces in the partly cracked case. To understand this we look at a simple one dimensional case as shown in Figure 2.

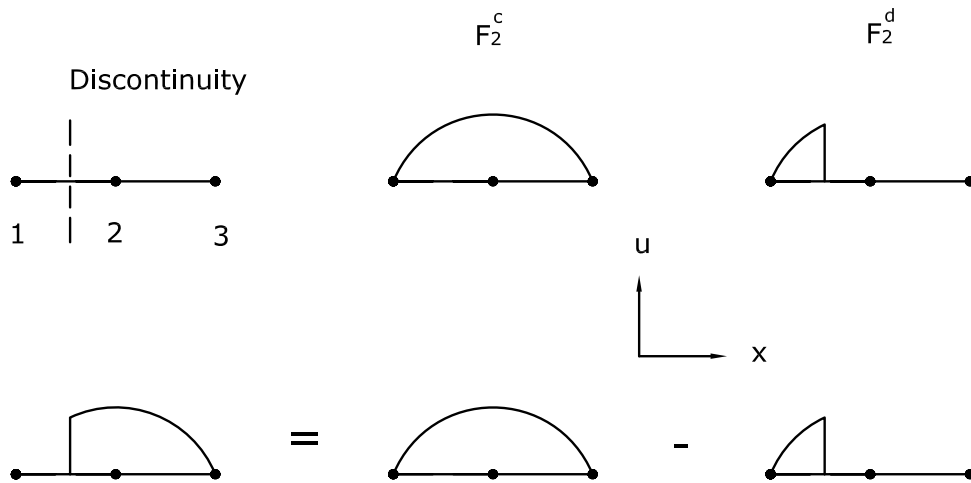


Figure 2: Continuous and discontinuous shape functions for the mid node in a quadratic line element

When introducing a continuous shape function and the corresponding discontinuous shape function it is essential to note that the discontinuous shape function only contributes on one side of the discontinuity. A combination with the continuous shape function will give a similar discontinuous contribution on the opposite side of the discontinuity as shown in the lower part of Figure 2. A discontinuous enrichment must always be followed by its corresponding continuous enrichment in order to fulfill the basic concept in XFEM (decoupling the field variable across the discontinuity). This is naturally fulfilled for the fully cracked standard elements, but not for the partly cracked elements, where the enrichment only is within a subtriangle of the element. Therefore continuous enrichments are needed in these subtriangles as well. The composition of the complete displacement field is shown in figure 3.

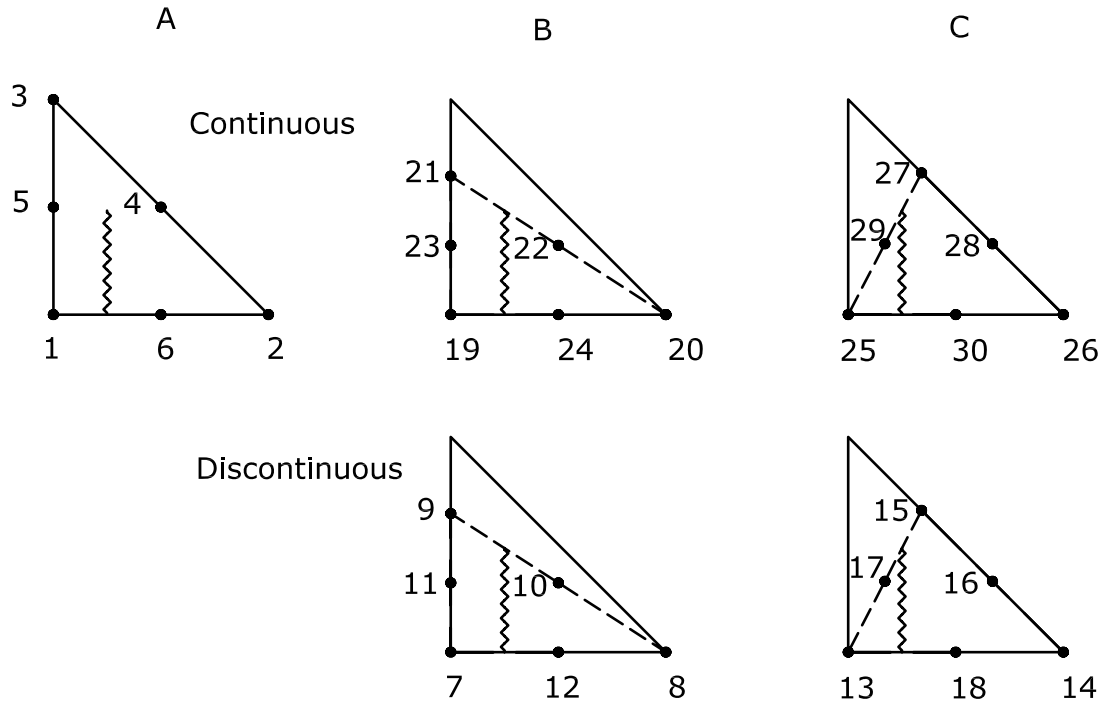


Figure 3: Nodes in the partly cracked tip element

Restrictions in the tip element

After introducing enrichments in the two subtriangles a set of restrictions must be set in order to fulfill compatibility in displacements within the element.

For the two continuous subtriangles, displacement contributions can only be allowed along the cracked edge. If displacements were allowed on the tip edge, the displacements would not be internally compatible. If displacement contributions were allowed on the uncracked element edge, the element would not be compatible with surrounding standard elements. Therefore only continuous contributions from the subtriangles are allowed from nodes [24 30] see Figure 3. The remaining 10 nodes must be eliminated or supported [19 20 21 22 23 25 26 27 28 29].

For the two discontinuous subtriangles the same compatibility restrictions must be fulfilled. Per definition there are no contributions on the uncracked edge due to the definition of the discontinuous shape functions. To fulfill the condition with a zero crack opening at the tip edge no displacement contribution can be allowed along side 1b opposite node 1b in subtriangle B and side 2c in subtriangle C. Therefore nodes [8 9 10 15 13 17] must be eliminated.

Side local shape functions

For the fully cracked element to couple easily to the partly cracked element all shape functions in the element are made element side local. Specifically it is only node 8 which gives contributions on more than one element side. This shape function is redefined so that it is element side local. The shape function from node 14 is identical to the shape function from node 8 along side 3 in the element triangle A. Subtracting 14 from 8 then gives a new element side local shape function.

$$F'_8 = F_8 - F_{14} \quad (3)$$

When the partly cracked element becomes fully cracked a number of dependencies arises in the displacement field. Analyzing the displacement field these dependencies can be found analytically and eliminated.

Coupling of nodes between elements

Along the connection side of the cracked elements the contribution from elements may be a sum of more than one element node. In order to make the coupling between the cracked element edges, a set of sum/difference nodes are constructed, so that the connection on system level becomes direct.

For the continuous nodes [6 24 30] a sum node is established, which then on system level can be coupled to the neighbour element. Further two difference nodes are established, which does not give contributions on the side i.e. the internal nodes.

$$\begin{aligned} F'_6 &= \frac{1}{3}(F_6 + F_{24} + F_{30}) \\ F'_{24} &= F_{24} - F_6 \\ F'_{30} &= F_{30} - F_6 \end{aligned} \quad (4)$$

For the discontinuous nodes [12 18] a sum and a difference node is established as well.

$$\begin{aligned} F'_{12} &= \frac{1}{2}(F_{12} + F_{18}) \\ F'_{18} &= F_{12} - F_{18} \end{aligned} \quad (5)$$

FEM EQUATIONS

The variational formulation of the equilibrium equations gives the work conjugated stresses and strains for the crack written together with an incremental form of the constitutive law for the cracks stress/stain relation.

$$\boldsymbol{\epsilon}_{cr} = \begin{bmatrix} \Delta u_n \\ \Delta u_s \end{bmatrix} \quad \boldsymbol{\sigma}_{cr} = \begin{bmatrix} \sigma_n \\ \tau_{ns} \end{bmatrix} \quad d\boldsymbol{\sigma}_{cr} = \mathbf{D}_T^{cr} d\boldsymbol{\epsilon}_{cr} \quad (6)$$

where σ_n and τ_{ns} is respectively the normal and the shear stress in the crack. $\Delta u_s = (u_n^+ - u_n^-)$ is the normal opening of the crack and $\Delta u_s = (u_s^+ - u_s^-)$ is the sliding of the crack. \mathbf{D}_T is the tangent material stiffness

In the continuum Ω the usual generalized stresses and strains are found. In the present work linear elasticity and plane stress is assumed.

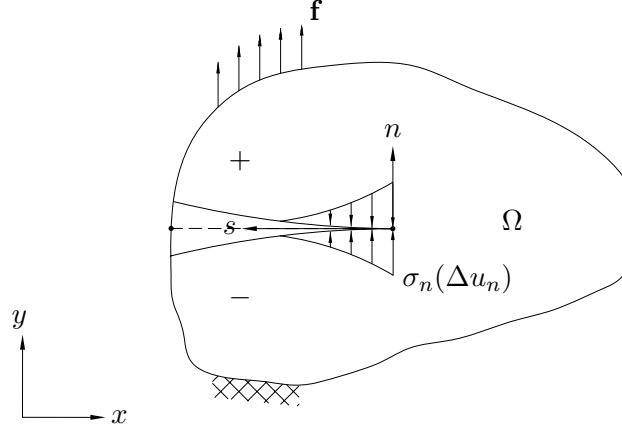


Figure 4: Cohesive crack model in 2D. The model shows the orientation of the crack with positive and negative side. The model is based on the principles of the Fictitious crack model by Hillerborg et al. (1976)

$$\boldsymbol{\epsilon}(x, y) = \begin{bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{bmatrix} = \mathbf{B}^c(x, y)\mathbf{v}^c + \mathbf{B}^d(x, y)\mathbf{v}^d \quad (7)$$

$$\boldsymbol{\epsilon}_{cr}(s) = \begin{bmatrix} \Delta u_n \\ \Delta u_s \end{bmatrix} = \mathbf{T} \begin{bmatrix} \Delta u_x \\ \Delta u_y \end{bmatrix} = \mathbf{T}(\mathbf{N}_+^d(s) - \mathbf{N}_-^d(s))\mathbf{v}^d = \mathbf{B}_{dd}(s)\mathbf{v}^d \quad (8)$$

where \mathbf{T} is a transformation matrix translating from global xy -coordinates to local crack coordinates ns . \mathbf{B}_{dd} is the crack strain distribution matrix.

The tangent stiffness

For the continuous part of the element the incremental virtual internal work can be formulated as

$$\begin{aligned} W_{cont}^i &= \int_{V_{cont}} \delta \boldsymbol{\epsilon}^T \boldsymbol{\sigma} dV \\ &= \delta \mathbf{v}^T \int_{V_{cont}} \begin{bmatrix} \mathbf{B}_c & \mathbf{B}_d \end{bmatrix}^T \mathbf{D}_T \begin{bmatrix} \mathbf{B}_c & \mathbf{B}_d \end{bmatrix} dV \mathbf{v} \\ &= \delta \mathbf{v}^T \begin{bmatrix} \int_{V_{cont}} \mathbf{B}_c \mathbf{D}_T \mathbf{B}_c dV & \int_{V_{cont}} \mathbf{B}_c \mathbf{D}_T \mathbf{B}_d dV \\ \int_{V_{cont}} \mathbf{B}_d \mathbf{D}_T \mathbf{B}_c dV & \int_{V_{cont}} \mathbf{B}_d \mathbf{D}_T \mathbf{B}_d dV \end{bmatrix} \mathbf{v} \\ &= \delta \mathbf{v}^T \begin{bmatrix} \mathbf{k}_T^{cc} & \mathbf{k}_T^{cd} \\ \mathbf{k}_T^{dc} & \mathbf{k}_T^{dd} \end{bmatrix} \mathbf{v} \end{aligned} \quad (9)$$

Where \mathbf{D}_T is the continuous material tangent stiffness, \mathbf{k}_T^{cc} is the element tangent stiffness contribution from continuous DOFs, \mathbf{k}_T^{dd} is the element tangent stiffness contribution from discontinuous DOFs and \mathbf{k}_T^{cd} and \mathbf{k}_T^{dc} are the element tangent stiffness contributions from the interaction between the continuous and discontinuous DOFs.

Analogously, the incremental virtual internal work done in the crack can be formulated as

$$\begin{aligned}
W_{cr}^i &= \int_{cr} \delta \epsilon_c r^T \sigma_{cr} ds \\
&= \delta \mathbf{v}^T \int_{cr} [\mathbf{0} \quad \mathbf{B}_{dd}]^T \mathbf{D}_T^{cr} [\mathbf{0} \quad \mathbf{B}_{dd}] ds \mathbf{v} \\
&= \delta \mathbf{v}^T \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \int_{cr} \mathbf{B}_{dd} \mathbf{D}_T^{cr} \mathbf{B}_{dd} ds \end{bmatrix} \mathbf{v} \\
&= \delta \mathbf{v}^T \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_T^{cr} \end{bmatrix} \mathbf{v}
\end{aligned} \tag{10}$$

where \mathbf{D}_T^{cr} is the crack tangent stiffness matrix and \mathbf{k}_T^{cr} is the element tangent stiffness contribution from the crack.

By summation the total incremental internal virtual work can be found

$$\begin{aligned}
W^i &= \delta \mathbf{v}^T \begin{bmatrix} \mathbf{k}_T^{cc} & \mathbf{k}_T^{cd} \\ \mathbf{k}_T^{dc} & \mathbf{k}_T^{dd} + \mathbf{k}_T^{cr} \end{bmatrix} \mathbf{v} \\
&= \delta \mathbf{v}^T \mathbf{K}_T \mathbf{v}
\end{aligned} \tag{11}$$

where \mathbf{K}_T is the element tangent stiffness matrix.

The inner nodal forces

The internal nodal forces \mathbf{Q} corresponding to a given displacement state \mathbf{v} has to be formulated in order to solve the nonlinear equations. In general this is done using the actual stress level σ which here is a function of the displacements. The only nonlinear contribution is from the cracked elements where the cohesive stresses in general have a nonlinear behaviour. The nodal forces can for an element be calculated as

$$\mathbf{q} = \int_{el} \mathbf{B}^T \sigma d\Omega \tag{12}$$

For the cracked elements this is formulated using the terms and organization from the previous section as

$$\mathbf{q} = \begin{bmatrix} \int_{el} \mathbf{B}_c^T \sigma d\Omega \\ \int_{el} \mathbf{B}_d^T \sigma d\Omega + \int_{cr} \mathbf{B}_{cr} \sigma_{cr} dS \end{bmatrix} = \begin{bmatrix} \mathbf{q}_c \\ \mathbf{q}_d \end{bmatrix} \tag{13}$$

When \mathbf{K}_T and \mathbf{Q} has been build-up a general nonlinear solution strategy e.g. arclength can be applied to solve the system equations.

CRACK PROPAGATION CRITERION

When applying the crack tip element evaluation of stresses at the crack tip is essential. Because of the geometrical position of the enrichment fields the stress evaluation at the crack tip is not unique. Figure shows a subtriangulation of the crack tip element. Due to the enrichment fields each of these subtriangles have a stress answer at the crack tip. Some kind of stress average is needed in order to have a unique tip stress answer. In the present work an average based on a weighting with the respective areas of subtriangles has been used.

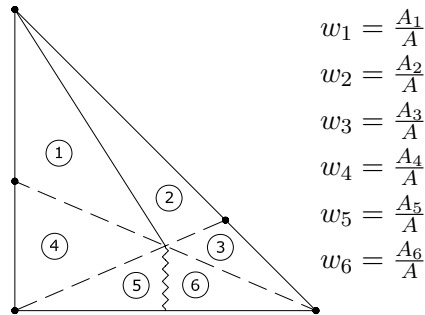


Figure 5: Subtriangulation of the crack tip element based on the enrichment geometry. Each of the six parts have their own stress answer at the crack tip. The figure indicates the weights $w_1 - w_6$ used when averaging the stresses at the crack tip.

In previous works a lot of effort have been put into stabilization of the crack growth. This has primarily been done using some non local stress averaging methods. In the present work the scope has been on developing a crack tip element which can represent better stresses at the crack tip. Therefore results presented in this work are based on stress evaluations in a single point just in front of the crack tip. I.e. the performance of the crack tip element is not influenced by a non local averaging in the surroundings of the crack tip.

The strategy used for crack propagation is based on principal stresses and directions. When the crack is about to enter an element the crack is set to propagate perpendicular to the largest principal stress direction. Within an element the crack cannot change direction, the crack propagation is therefore straight lines through the element mesh.

EXAMPLE

As an example a sheet containing an initial flaw is considered. A semi analytical solution exist if the sheet can be assumed infinite Dick-Nielsen et al. (2006). This semi analytical solution will in the following be used for verification of the model. Figure 6 shows the bilinear cohesive law for the applied material and the geometry of the flaw. The surrounding material is assumed linear elastic with plane stress conditions.

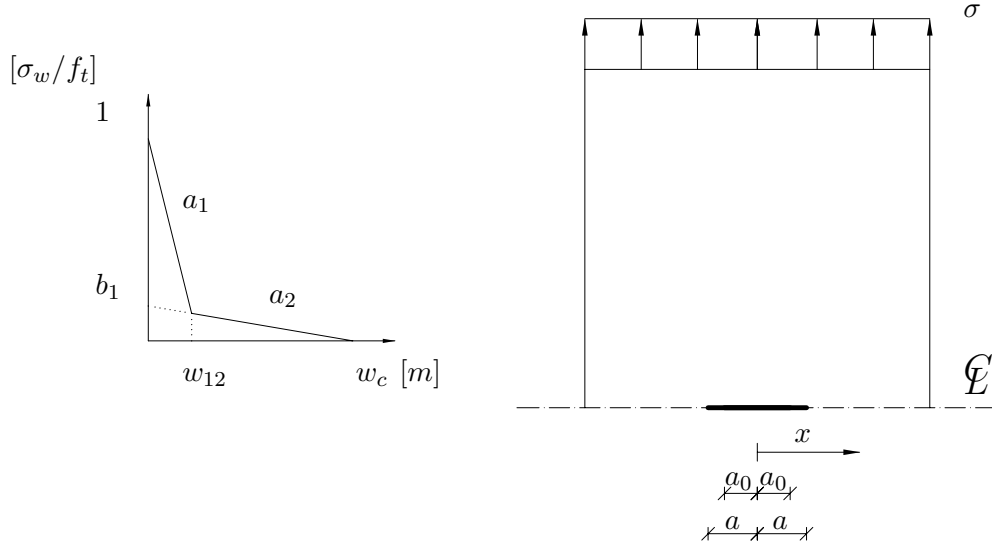


Figure 6: a) The bilinear cohesive law used b) The geometry of flaw and coordinate system

The following material data were used in this example $f_t = 2.83$ MPa, $a_1 = 156$ mm⁻¹, $a_2 = 9.7$ mm⁻¹ and $b_2 = 0.24$ this correspond to a fracture energy $G_f = 14.05$ J/m². For the elastic properties of the surrounding material were used an Young's modulus $E = 31$ GPa and a Poissons's ratio $\nu = 0.2$. An initial flaw with length of 4 mm ($a_0 = 2$ mm) were used. To fulfill the infinite conditions satisfactorily dimensions of the sheet are 1200 by 1000 mm (w x h). In the region around the flaw an element size of approximate 4mm has been used see Figure 8, this is approximately 10% of the fracture process zone ($l_p \sim EG_f/f_t^2 \sim 50$ mm). Results are presented in figure 7. The graph shows the far-field stress σ versus the half crack length a obtained in the sheet. Both the XFEM solution with partly cracked elements and the semi analytical solution are presented in this graph. From the graph it is seen that the overall response is captured on a smooth curve. With this discretization the error is kept within 5% compared to the semianalytical solution.

The small discrepancy seen in this example is believed to be caused by to few elements along the relatively short fracture process zone for the applied mortar.

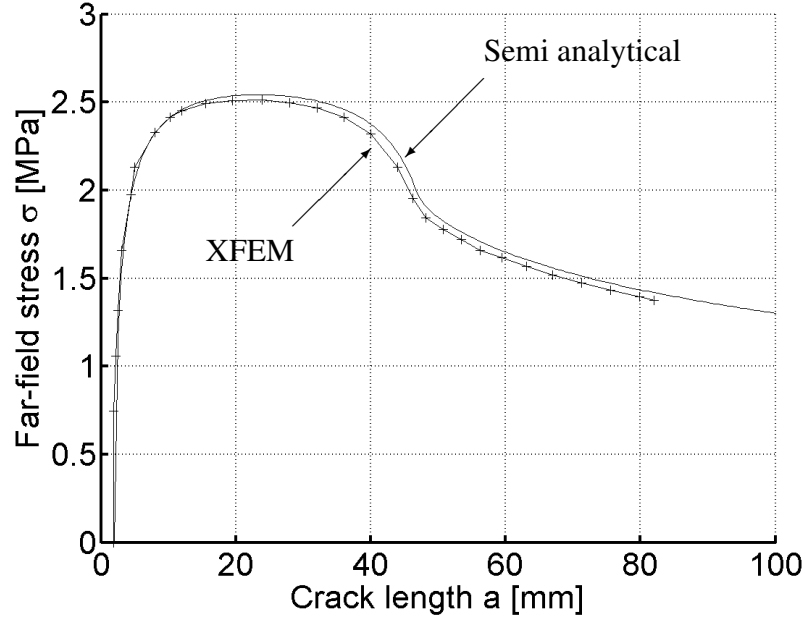


Figure 7: The far-field stress σ plotted versus the half-crack length a for a XFEM solution with the present crack tip formulation. For comparison the semi analytical solution obtained by Dick-Nielsen et al. (2006) is plotted in the same graph.

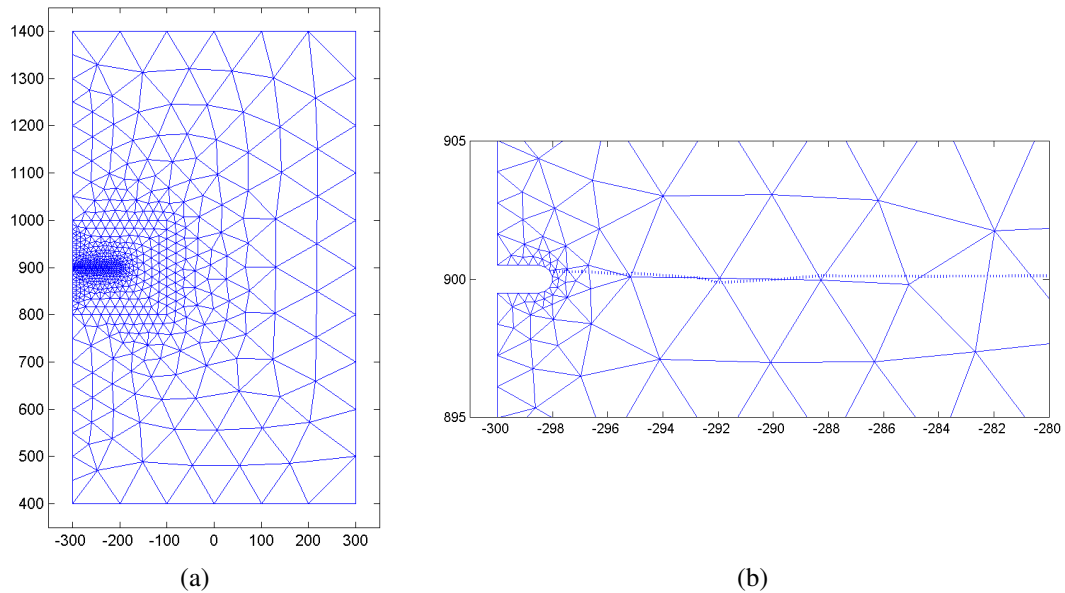


Figure 8: a) Overall mesh of the 'infinite' sheet dimensions are 1200 by 1000 mm (w x h) b) Zoomed view of the mesh near the flaw, where the crack is seen to propagate freely through the mesh starting at the flaw

CONCLUSION

In the present work a cohesive crack tip element has been developed based on the principles of the eXtended Finite Element Method. The element is based on a double enriched displacement field of Linear Strain Triangle type. This double enrichment makes the element capable of reproducing equal stresses at both sides of the crack at the tip. In order to achieve a complete enrichment on both sides of the crack both a discontinuous and a continuous displacement field is used as enrichment in the two enrichment triangles. Due to these new enrichments the stress accuracy within the crack tip element has been improved to a level, so that the crack tip stresses can be evaluated locally without introducing nonlocal averaging in the surroundings of the crack tip. The crack tip element can be used as a fully cracked element, when a few dependencies are eliminated. As example an infinite sheet in pure tension with an initial flaw has been considered. The results from the XFEM computation using the crack tip element is compared with a semi analytical solution of the problem, and shows a good agreement.

REFERENCES

- J. L. Asferg, P. N. Poulsen, and L. O. Nielsen. A direct xfem formulation for modeling of cohesive crack growth in concrete. *Computers and Concrete*, 4(2):83–100, 2007. ISSN 15988198.
- T. Belytschko and T. Black. Elastic crack growth in finite elements with minimal remeshing. *International Journal for Numerical Methods in Engineering*, (45):601–620, 2003.
- Lars Dick-Nielsen, Henrik Stang, and Peter Noe Poulsen. Condition for strain-hardening in ecc uniaxial test specimen. In *Measuring, Monitoring and Modeling Concrete Properties*, pages 41–47, Netherlands, 2006. Springer. ISBN 1-4020-5103-4. Presented at: Measuring, Monitoring and Modeling Concrete Properties : In Honor of Surendra P. Shah, MMMCP ; 1 : Alexandroupolis, Greece - Democritus University of Thrace, 2006.
- A. Hillerborg, M. Mod  er, and P-E. Peterson. Analysis of crack formation and crack growth in concrete by means of fracture mechanics and finite elements. *Cem. Concr. Res.*, (6):773–782, 1976.
- Goangseup Zi and Ted Belytschko. New crack-tip elements for xfem and applications to cohesive cracks. *International Journal for Numerical Methods in Engineering*, 57(15):2221–2240, 2003. ISSN 00295981.